

# Universal Fractional Map and Cascade of Bifurcations Type Attractors

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We modified the way in which the Universal Map is obtained in the regular dynamics to derive the Universal  $\alpha$ -Family of Maps depending on a single parameter  $\alpha > 0$  which is the order of the fractional derivative in the nonlinear fractional differential equation describing a system experiencing periodic kicks. We consider two particular  $\alpha$ -families corresponding to the Standard and Logistic Maps. For fractional  $\alpha < 2$  in the area of parameter values of the transition through the period doubling cascade of bifurcations from regular to chaotic motion in regular dynamics corresponding fractional systems demonstrate a new type of attractors - cascade of bifurcations type trajectories.

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Fractional differential equations (FDE) are frequently used in science and engineering to describe systems with memory (see [1, 2] and references there). Circuit elements with memory: memristors, memcapacitors, and meminductors (see Wikipedia [3]) can be used to model such systems. Nonlinearity plays an important role in such systems [4]. We'll call systems which can be described by the FDE fractional dynamical systems (FDS) even though the group property is not satisfied in this case and the values of the system variables depend on the whole history of the system's evolution. Because FDE are integro-differential equations and there are no high order numerical algorithms to simulate such equations, derivation of the fractional maps (FM) is important for the investigation of the general properties of the FDS. First FM were derived from the FDE in [5–8]. First results of the investigation of the FM (see [6, 7, 9, 10]) revealed new properties of the FDS. Cascade of bifurcations type trajectories (CBTT) are the most unusual features of the investigated FM. In CBTT cascade of bifurcations is not a result of the change in a system parameter but appears as the attracting single trajectory and is a new type of attractors. All previous investigations of the FM were done on the various forms of the fractional 2-dimensional Standard Map corresponding to the order  $1 < \alpha \leq 2$  of the fractional derivative. CBTT appeared in all investigated FM. The role of the cascades of bifurcations in the transition from order to chaos in the regular dynamics and their connection to the scaling properties of the corresponding systems are well investigated (see [11]). Consideration of the origin and the necessary and sufficient conditions of the CBTT's existence requires further investigation of the FM, which includes development of the simple, if possible one-dimensional, FM. The best investigated one-dimensional regular map is the ubiquitous Logistic Map. This map, and the maps with  $\alpha \leq 1$  in general, can't be derived in a way previously used ([1, 8]) to derive FM for  $\alpha > 1$  (for the discussion on the way in which the Logistic Map can be derived from a differential

equation see [12]).

To derive the equations of the Universal  $\alpha$ -Family of Maps (U $\alpha$ FM) let's consider the following equation:

$$\frac{d^\alpha x}{dt^\alpha} + G_K(x(t - \Delta T)) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - (n + \varepsilon)\right) = 0, \quad (1)$$

where  $\varepsilon > \Delta > 0$ ,  $\alpha \in \mathbb{R}$ ,  $\alpha > 0$ , in the limit  $\varepsilon \rightarrow 0$ . The initial conditions should correspond to the type of the fractional derivative we are going to use. In the case  $\alpha = 2$ ,  $\Delta = 0$ , and  $G_K(x) = KG(x)$  Eq. (1) corresponds to the equation which integration produces the regular Universal Map. Case  $\Delta = 0$ ,  $G_K(x) = KG(x)$ ,  $\alpha > 1$  has been used to derive the fractional Universal Map.  $\Delta \neq 0$  is essential for the case  $\alpha \leq 1$  when  $x(t)$  is a function discontinued at the time of the kicks [5, 12] and the use of the  $K$  as a parameter rather than a factor is necessary to extend the class of the considered maps to include the Logistic Map. Without losing the generality we assume  $T = 1$ . Case  $T \neq 1$  is considered in [12] and can be reduced to this case by rescaling the time variable.

In the case of the Riemann-Liouville fractional derivative Eq. (1) can be written as

$${}_0D_t^\alpha x(t) + G_K(x(t - \Delta)) \sum_{n=-\infty}^{\infty} \delta(t - (n + \varepsilon)) = 0, \quad (2)$$

where  $\varepsilon > \Delta > 0$ ,  $\varepsilon \rightarrow 0$ ,  $0 \leq N - 1 < \alpha \leq N$ ,  $\alpha \in \mathbb{R}$ ,  $N \in \mathbb{Z}$ , and the initial conditions  $({}_0D_t^{\alpha-k} x)(0+) = c_k$ ,  $k = 1, \dots, N$ . The left-sided Riemann-Liouville fractional derivative  ${}_0D_t^\alpha x(t)$  defined for  $t > 0$  [13–15] as

$$\begin{aligned} {}_0D_t^\alpha x(t) &= D_t^n {}_0I_t^{n-\alpha} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{\alpha-n+1}}, \end{aligned} \quad (3)$$

where  $n - 1 \leq \alpha < n$ ,  $n \in \mathbb{Z}$ ,  $D_t^n = d^n/dt^n$ , and  ${}_0I_t^\alpha$  is a fractional integral.

This problem can be reduced [1, 15, 16] to the Volterra

integral equation of the second kind for  $t > 0$

$$x(t) = \sum_{k=1}^N \frac{c_k}{\Gamma(\alpha - k + 1)} t^{\alpha-k} - \frac{1}{\Gamma(\alpha)} \int_0^t d\tau \frac{G_K(x(\tau - \Delta))}{(t - \tau)^{1-\alpha}} \sum_{k=-\infty}^{\infty} \delta(\tau - (k + \varepsilon)), \quad (4)$$

which integration gives ( $t > 0$ )

$$x(t) = \sum_{k=1}^{N-1} \frac{c_k}{\Gamma(\alpha - k + 1)} t^{\alpha-k} - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[t-\varepsilon]} \frac{G_K(x(k + \varepsilon - \Delta))}{(t - (k + \varepsilon))^{1-\alpha}} \Theta(t - (k + \varepsilon)), \quad (5)$$

where  $\Theta(t)$  is the Heaviside step function. In Eq. (5) we took into account that boundness of  $x(t)$  at  $t = 0$  requires  $c_N = 0$  and  $x(0) = 0$ .

With the introduction [5]  $p(t) = {}_0D_t^{\alpha-N+1}x(t)$ ,  $p^{(s)}(t) = D_t^s p(t)$ ,  $s = 0, 1, \dots, N-2$  Eq. (5) leads to

$$p^{(s)}(t) = \sum_{k=1}^{N-s-1} \frac{c_k}{(N-s-1-k)!} t^{N-s-1-k} - \frac{1}{(N-s-2)!} \sum_{k=0}^{[t-\varepsilon]} G_K(x(k + \varepsilon - \Delta)) (t - k)^{N-s-2}, \quad (6)$$

where  $s = 0, 1, \dots, N-2$ . With the definitions  $x_n = x(n)$  and  $p_n^{(s)} = p^{(s)}(n)$  Eqs. (5) and (6) in the limit  $\varepsilon \rightarrow 0$  give for  $t=n+1$  the Riemann-Liouville U $\alpha$ FM (U $\alpha$ RLFM)

$$x_{n+1} = \sum_{k=1}^{N-1} \frac{c_k}{\Gamma(\alpha - k + 1)} (n+1)^{\alpha-k} - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n G_K(x_k) (n - k + 1)^{\alpha-1}, \quad (7)$$

$$p_{n+1}^s = \sum_{k=1}^{N-s-1} \frac{c_k}{(N-s-1-k)!} (n+1)^{N-s-1-k} - \frac{1}{(N-s-2)!} \sum_{k=0}^n G_K(x_k) (n - k + 1)^{N-s-2}. \quad (8)$$

Map equations for the momentum defined in a usual way

$$p(t) = D_t^1 x(t), \quad p^s(t) = D_t^s p(t), \quad s = 0, 1, \dots, N-2, \quad (9)$$

and the discussion on the different ways of the defining momentum in the case of the Riemann-Liouville maps can be found in [12]. U $\alpha$ RLFM Eqs. (7) (8) can be writ-

ten in the much simpler form

$$p_{n+1}^s = p_n^s + \sum_{k=0}^{N-s-3} \frac{p_n^{k+s+1}}{(k+1)!} - \frac{G_K(x_n)}{(N-s-2)!}; \quad (10)$$

$$x_{n+1} = \sum_{k=2}^{N-1} \frac{c_k}{\Gamma(\alpha - k + 1)} (n+1)^{\alpha-k} + \frac{1}{\Gamma(\alpha)} p_{n+1}^{N-2} + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} p_{k+1}^{N-2} V_\alpha^1(n - k + 1), \quad (11)$$

where  $s = 0, 1, \dots, N-2$  and  $V_\alpha^k(m) = m^{\alpha-k} - (m-1)^{\alpha-k}$ .

For the integer  $\alpha = N$  the U $\alpha$ FM converges to

$$p_{n+1}^s = p_n^s + \sum_{k=0}^{N-s-3} \frac{p_n^{k+s+1}}{(k+1)!} - \frac{G_K(x_n)}{(N-s-2)!}; \quad (12)$$

$$x_{n+1} = x_n + \sum_{k=0}^{N-2} \frac{p_n^k}{(k+1)!} - \frac{G_K(x_n)}{(N-1)!} \quad (13)$$

with the Jacobian ( $N \times N$ ,  $N \geq 2$ )

$$\begin{vmatrix} 1 - \frac{\dot{G}_K(x)}{\Gamma(N)} & 1 & \frac{1}{2} & \dots & \frac{1}{\Gamma(n)} & \dots & \frac{1}{\Gamma(N-1)} & \frac{1}{\Gamma(N)} \\ -\frac{\dot{G}_K(x)}{\Gamma(N-1)} & 1 & 1 & \dots & \frac{1}{\Gamma(n-1)} & \dots & \frac{1}{\Gamma(N-2)} & \frac{1}{\Gamma(N-1)} \\ -\frac{\dot{G}_K(x)}{\Gamma(N-2)} & 0 & 1 & \dots & \frac{1}{\Gamma(n-2)} & \dots & \frac{1}{\Gamma(N-3)} & \frac{1}{\Gamma(N-2)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{\dot{G}_K(x)}{\Gamma(N-k+1)} & 0 & 0 & \dots & \frac{1}{\Gamma(n-k+1)} & \dots & \frac{1}{\Gamma(N-k)} & \frac{1}{\Gamma(N-k+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\dot{G}_K(x) & 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{vmatrix},$$

where  $n$  and  $k$  are the column and row numbers. The first column can be written as the sum of the column with one in the first row and the remaining zeros and the column which is equal to  $\dot{G}_K(x)$  times the last column. Determinants of the corresponding matrices are 1 and 0, this is why the Jacobian is equal to one and the map is the  $N$ -dimensional volume preserving map.

The integer U $\alpha$ FM's fixed points are  $p_0^s = 0$  ( $s = 0, \dots, N-2$ ) and  $x_0$  from  $G(x_0) = 0$ . Their stability for  $N \geq 1$  is defined by the eigenvalues  $\lambda$  of the Jacobian matrix  $J(x_0, p_0^0, \dots, p_0^{N-2})$ . Polynomial  $P(\lambda) = \det[J(x_0, p_0^0, \dots, p_0^{N-2}) - \lambda I]$  has values  $P(0) = \lambda_1 \times \dots \times \lambda_N = 1$  and  $P(1) = (-1)^N \dot{G}_K(x_0)$ , which means that for odd values of  $N > 1$  stability is possible only if  $\dot{G}_K(x_0) = 0$ . For  $T = 2$  points  $p_{n+1}^s = -p_n^s$  ( $s = 0, \dots, N-2$ ) and  $G(x_{n+1}) = -G(x_n)$ . In the case  $N = 3$  the only  $T = 2$  points are the periodic points.

For Eq. (1) with the left-sided Caputo derivative [15]

$$\begin{aligned} {}^C_0 D_t^\alpha x(t) &= I_t^{n-\alpha} D_t^n x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{D_\tau^n x(\tau) d\tau}{(t-\tau)^{\alpha-n+1}} \quad (n-1 < \alpha \leq n) \end{aligned} \quad (14)$$

the initial conditions may be taken as  $(D_t^k x)(0+) = b_k$ ,  $k = 0, \dots, N-1$ . This problem is equivalent to the

Volterra integral equation of the second kind ( $t > 0$ )

$$x(t) = \sum_{k=0}^{N-1} \frac{b_k}{k!} t^k - \frac{1}{\Gamma(\alpha)} \int_0^t d\tau \frac{G_K(x(\tau - \Delta))}{(t - \tau)^{1-\alpha}} \sum_{k=-\infty}^{\infty} \delta(\tau - (k + \varepsilon)). \quad (15)$$

With the introduction  $x^{(s)}(t) = D_t^s x(t)$  the Caputo U $\alpha$ FM (U $\alpha$ CFM) can be derived in the form [1]

$$x_{n+1}^{(s)} = \sum_{k=0}^{N-s-1} \frac{x_0^{(k+s)}}{k!} (n+1)^k - \frac{1}{\Gamma(\alpha-s)} \sum_{k=0}^n G_K(x_k) (n-k+1)^{\alpha-s-1}, \quad (16)$$

where  $s = 0, 1, \dots, N-1$ .

Fractional maps Eqs. (10), (11), and (16) are maps with memory in which the next value of the map variables depends on all previous values. An increase in  $\alpha$  leads to the increase in the dimension of the map and to the power law increase of the weights of the old states (the increased role of memory). Integer values of  $\alpha$  correspond to the degenerate cases in which map equations can be written as maps with full memory [17] which are equivalent to the one step memory maps (for a discussion on the fractional maps as maps with memory see [12]).

In the  $\alpha = 2$  case Eqs. (12) and (13) produce the Standard Map if  $G_K(x) = K \sin(x)$  and in the  $\alpha = 1$  case the Logistic Map results from  $G_K(x) = x - Kx(1-x)$ . We'll call the U $\alpha$ FM Eqs. (10) and (11) with  $G_K(x) = K \sin(x)$  the Standard  $\alpha$ -RL-Family of Maps (S $\alpha$ RLFM) and with  $G_K(x) = x - Kx(1-x)$  the Logistic  $\alpha$ -RL-Family of Maps (L $\alpha$ RLFM); we'll call U $\alpha$ FM Eq. (16) with  $G_K(x) = K \sin(x)$  the Standard  $\alpha$ -Caputo-Family of Maps (S $\alpha$ CFM) and with  $G_K(x) = x - Kx(1-x)$  the Logistic  $\alpha$ -Caputo-Family of Maps (L $\alpha$ CFM).

For  $\alpha = 0$  the solution of Eq. (1) is identical zero. For  $\alpha < 1$  the U $\alpha$ RLFM Eq. (7) also produces identical zero for maps which satisfy  $G(0) = 0$ , which is true for the S $\alpha$ RLFM and L $\alpha$ RLFM. The  $\alpha = 1$  S $\alpha$ RLFM is a particular form of the Circle Map with zero driving phase

$$x_{n+1} = x_n - K \sin(x_n), \quad (\text{mod } 2\pi) \quad (17)$$

(see its bifurcation diagram Fig. 1(a)). In this map the stable  $T = 4$  sink appears at  $K \approx 3.445$  and the transition to chaos through the period doubling cascade of bifurcations occurs at  $K \approx 3.532$  [12]. In the  $\alpha = 2$  Standard Map

$$p_{n+1} = p_n - K \sin x, \quad x_{n+1} = x_n + p_{n+1} \quad (18)$$

$T = 4$  stable elliptic points appear at  $K \approx 6.59$  and the period doubling cascade of bifurcations leads to the disappearance of the islands of stability in the chaotic sea at

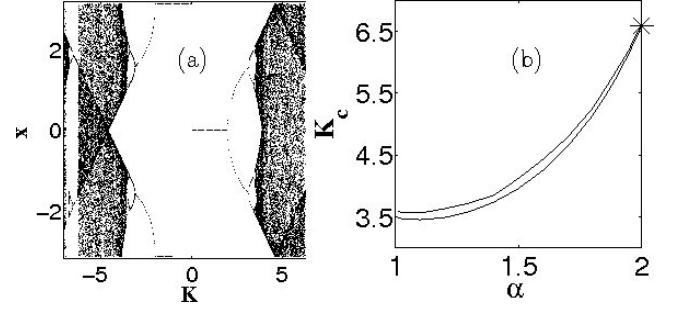


FIG. 1. (a) The bifurcation diagram for 1D Standard Map Eq. 17. (b) On  $K - \alpha$  graph in S $\alpha$ RLFM and S $\alpha$ CFM CBTT exist in the band of the map's parameters ending at the cusp in the top right corner. Star marks the point at which the Standard Map's ( $\alpha = 2$ )  $T = 2$  elliptic points with  $x_{n+1} = x_n - \pi$  and  $p_{n+1} = -p_n$  become unstable and bifurcate.

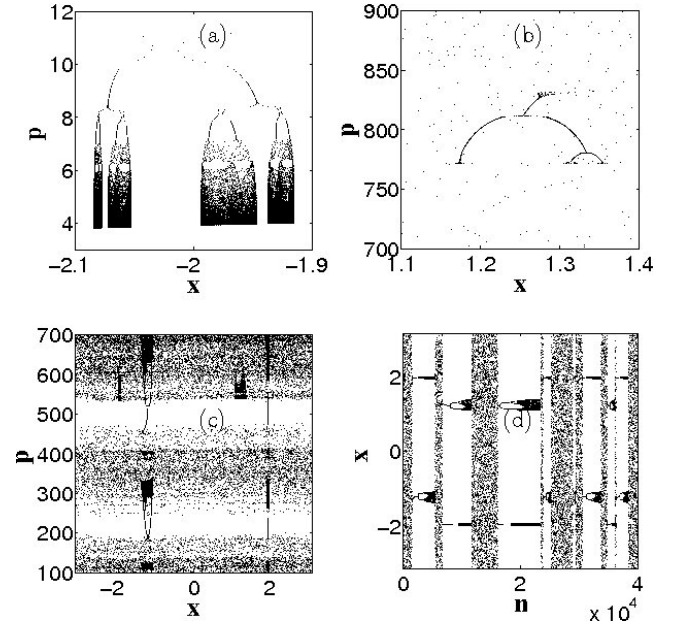


FIG. 2. A single CBTT in the S $\alpha$ RLFM. (a) One of the two branches of the CBTT for  $\alpha = 1.1, K = 3.5$ . (b) A zoom of a small feature in an intermittent trajectory for  $\alpha = 1.95, K = 6.2$ . (c) An intermittent trajectory in phase space for  $\alpha = 1.65, K = 4.5$ . (d)  $x$  of  $n$  for the case (c).

$K \approx 6.6344$  [18]. In S $\alpha$ RLFM and S $\alpha$ CFM for  $1 < \alpha < 2$  CBTT exist in the band between two curves connecting the above-mentioned points Fig. 1(b). The lower curve is calculated semi-analytically as the condition of the appearance of  $T = 4$  sink in the limit  $n \rightarrow \infty$ . Both curves are confirmed by the large number of computer simulations [9, 10]. Within the CBTT band trajectories evolve from being very stable features which exist for the longest time we were running our codes, 500000 iterations, when  $\alpha$  is close to one Fig. 2(a) to being barely distinguishable and short-lived features when  $\alpha$  is close to two Fig. 2(b). For the intermediate values of  $\alpha$  CBTT behave similar to the sticky trajectories in Hamiltonian dynamics: oc-

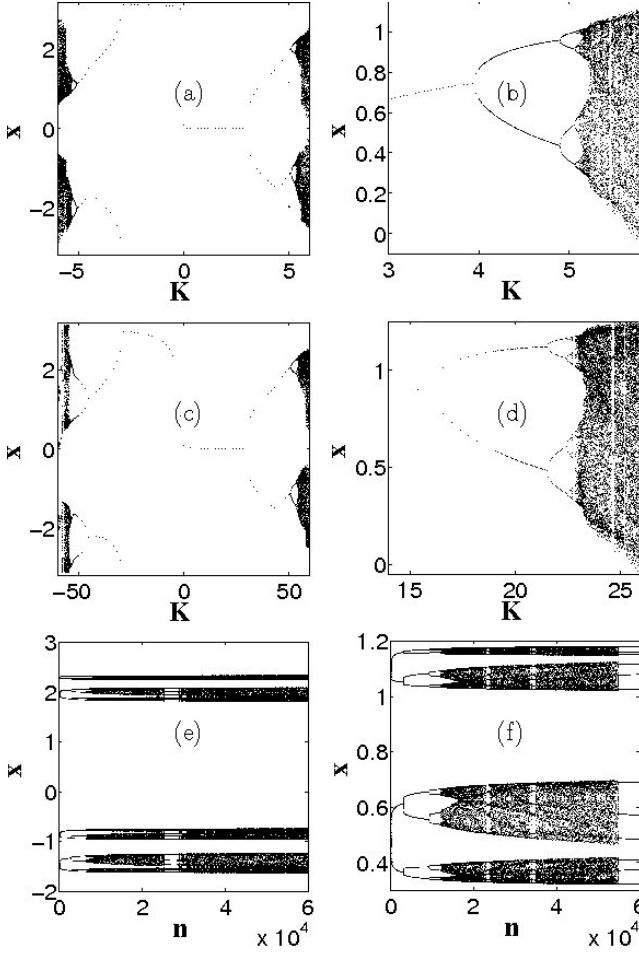


FIG. 3. Bifurcations and CBTT in  $S\alpha CFM$  and  $L\alpha CFM$  with  $0 < \alpha < 1$ . (a)-(d) bifurcation diagrams obtained after performing  $10^4$  iterations on a single trajectory with  $x_0 = 0.1$  for various values of  $K$ . (a)  $S\alpha CFM$  with  $\alpha = 0.5$ . (b)  $L\alpha CFM$  with  $\alpha = 0.5$ . (c)  $S\alpha CFM$  with  $\alpha = 0.05$ . (d)  $L\alpha CFM$  with  $\alpha = 0.1$ . (e) A CBTT in  $S\alpha CFM$  with  $\alpha = 0.01$  and  $K = 276$ . (f) A CBTT in  $L\alpha CFM$  with  $\alpha = 0.1$  and  $K = 22.7$ .

casional trajectories enter CBTT and then leave them entering the chaotic sea (Figs. 2(c),(d)). More on the properties of the  $\alpha = 1$  Standard Map and the consistency of the properties of the  $S\alpha RLFM$  and  $S\alpha CFM$  for  $1 \leq \alpha \leq 2$  can be found in [12].

There are no stable fixed points in the  $\alpha = 3$  Standard Map. For  $K^2 - 16 < 4p^{12} < K^2$  there exist two lines of the stable on the torus  $T = 2$  points. For more on the preliminary results of the investigation of the  $S\alpha RLFM$  and  $S\alpha CFM$  for  $2 < \alpha \leq 3$  see [12]. A different form of the 3D Standard Map has been recently introduced and investigated in [19].  $S\alpha RLFM$  and  $S\alpha CFM$  for  $\alpha > 2$  are poorly investigated and 3D volume preserving maps are not fully investigated.

With the corresponding  $G_K(x)$   $U\alpha CFM$  for  $0 < \alpha < 1$

$$x_{n+1} = x_0 - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n G(x_k)(n-k+1)^{\alpha-1}. \quad (19)$$

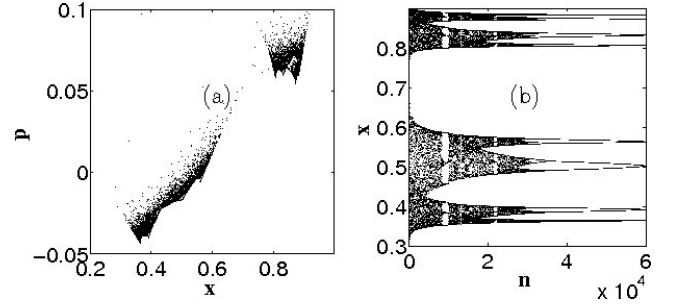


FIG. 4. An inverse CBTT in the  $L\alpha RLFm$  with  $\alpha = 1.15$ ,  $K = 3.45$ . 60000 iterations on a trajectory with  $x_0 = 0.01$  and  $p_0 = 0.1$ . (a) Phase space. (b)  $x-n$  graph.

produces  $S\alpha CFM$  and  $L\alpha CFM$ , which are one dimensional maps with the power law decreasing memory. The bifurcation diagrams for these maps are similar to the corresponding diagrams for the  $\alpha = 1$  case but are stretched along the parameter  $K$ -axis and the stretchiness increases with the decrease in  $\alpha$  Figs. 3(a)-(d). In the area of the parameter values for which on the bifurcation diagram stable periodic  $T > 2$  points exist individual trajectories are CBTT Figs. 3(e),(f).

The area preserving quadratic  $\alpha = 2$  Logistic Map

$$p_{n+1} = p_n + Kx_n(1-x_n) - x_n, \quad x_{n+1} = x_n + p_{n+1} \quad (20)$$

is similar to the map studied by Hénon [20] (for a recent review see [21]). The map Eq. (20) has two fixed points:  $(0, 0)$  stable for  $K \in (-3, 1)$  and  $((K-1)/K, 0)$  stable for  $K \in (1, 5)$ . At  $K = 5$  as a result of a bifurcation a couple of  $T = 2$  stable points appear. For more on the  $\alpha = 2$  and  $\alpha = 3$  quadratic Logistic Maps see [12]. 3D quadratic volume preserving maps were investigated in [22]. Logistic families of maps for  $1 < \alpha \leq 3$  are poorly investigated. Results of preliminary simulations show that CBTT are present in those maps. Fig. 4 shows an inverse CBTT in  $L\alpha RLFm$  with  $\alpha = 1.15$ ,  $K = 3.45$ .

*Summary.* The Universal  $\alpha$ -Family of Maps introduced in this paper is the extension of the fractional Universal Map, which allows consideration of the Logistic Map as its particular form. Using the Standard and Logistic Families of Maps we showed that the existence of the cascade of bifurcations type trajectories is a general property of the fractional dynamical systems. They appear for the parameter values corresponding to the transition through the period doubling cascade of bifurcations from regular to chaotic motion in the regular dynamics. Fig. 2 and Fig. 3 support our statement that with the increase in  $\alpha$ , which represents the increase in the systems' dimension and memory (increase in the weights of the earlier states), systems demonstrate more complex and chaotic behavior. Biological systems are systems with memory and the Fractional Logistic Map can serve as a basic model in population biology with memory. New types of materials with memory, such as memristors, memcapacitors,

and meminductors, can be used to model fractional systems to demonstrate the existence of the CBTT.  $\alpha > 2$  Standard and Logistic Maps (including their integer volume preserving forms) are topics of the ongoing research and their further investigation is necessary to demonstrate the consistency of the changes in the properties of the fractional systems with the change in  $\alpha$ .

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